E9 203 Compressed Sensing & Sparse SP

28 Apr. 2021

Last time:

- Analysis of BP via ℓ_1-coherence
- Analysis of thresholding-based algos. via ℓ_1-coherence.

Today:

- The restricted isometry property.

Chapter 6: The restricted isometry property (RIP)

Defn. The s^{th} restricted isometry constant (RIC) $\delta_s = \delta_s(A)$ of $A \in \mathbb{C}^{m\times N}$ is the smallest $\delta \geq 0$ s.t.

$$(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2 \quad — ①$$

for all s-sparse vecs. $x \in \mathbb{C}^N$.

Equivalently,

$$\delta_s = \max_{\substack{S \subset [N] \\ |S| \leq s}} \|A_S^H A_S - \underbrace{I}_{|S|\times|S|}\|_{2\to 2} \quad — ②$$

$\left(\|B\|_{2\to 2} = \max_{x\neq 0} \frac{\|Bx\|_2}{\|x\|_2}\right.$. We called this $|||B|||_2$ in the matrix theory class.$\left.\right)$

Note that ① is equivalent to

$$\left| \|A_S x\|_2^2 - \|x\|_2^2 \right| \leq \delta \|x\|_2^2 \text{ for all}$$

$S \subset [N]$, $|S| \leq s$, all $\underbrace{x \text{ s.t. } supp(x) = S}_{x \in \mathbb{C}^S}$

Now $A_S^H A_S - I$ is Hermitian, so we have, for $x \in \mathbb{C}^S$,

$$\|A_S x\|_2^2 - \|x\|_2^2 = \langle A_S x, A_S x\rangle - \langle x, x\rangle$$
$$= \langle (A_S^H A_S - I)x, x\rangle$$

$\langle A_S x, A_S x\rangle - \langle x,x\rangle$
$= \langle A_S^H A_S x, x\rangle - \langle x,x\rangle$
$= \langle (A_S^H A_S - I)x, x\rangle$.

$$\Rightarrow \max_{x \in \mathbb{C}^S\setminus\{0\}} \frac{\langle (A_S^H A_S - I)x, x\rangle}{\|x\|_2^2} = \|A_S^H A_S - I\|_{2\to 2}$$

Thus, ① is equivalent to

$$\max_{\substack{S \subset [N] \\ |S| \leq s}} \|A_S^H A_S - I\|_{2\to 2} \leq \delta,$$ i.e., ②, as

δ_s is the smallest such constant.

Remarks:

1. $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_N$
2. Although $\delta_s \geq 1$ is possible, the interesting case $\delta_s < 1$. Indeed, $\delta_s = \max_{\substack{S\subset[N] \\ |S|\leq s}} \|A_S^H A_S - I\|_{2\to 2}$

   $\Rightarrow$ all SVals of $A_S \in [1-\delta_s, 1+\delta_s]$, hence, if $\delta_s < 1$,

   $\Rightarrow$ A_S is injective.
3. In fact, $\delta_{2s} < 1$ is more relevant, as it implies $\|A(x-x')\|_2^2 > 0$ for all distinct s-sparse x, x'.

   $\Rightarrow$ distinct s-sparse vecs. map to distinct meas. vecs.

Prop. 6.2 If the matrix A has ℓ_2 normalized cols $a_1, a_2, \ldots, a_N$, (i.e., $\|a_i\|_2^2 = 1$, $i \in [N]$), then

$$\delta_1 = 0, \quad \delta_2 = \mu, \quad \delta_s \leq \mu_1(s-1) \leq (s-1)\mu, \quad s \geq 2.$$

Proof: Since the cols. of A are ℓ_2-normalized,

$$\|Ae_j\|_2^2 = \|e_j\|_2^2 \quad \forall j \in [N]$$

$\left[(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2 \quad \forall \underset{x = c\,e_j}{\text{1-sparse } x} \right]$

$\Rightarrow \delta_1 = 0.$

$$\delta_2 = \max_{\substack{S\subset[N] \\ |S|\leq 2}} \|A_S^H A_S - I\|_{2\to 2}$$
$$= \max_{1\leq i\neq j\leq N} \|A_{\{i,j\}}^H A_{\{i,j\}} - I\|_{2\to 2}$$

$$A_{\{i,j\}}^H A_{\{i,j\}} = \begin{bmatrix} 1 & \langle a_j, a_i\rangle \\ \langle a_i, a_j\rangle & 1 \end{bmatrix}$$

EVals of $(A_{\{i,j\}}^H A_{\{i,j\}} - I)$ are $|\langle a_i, a_j\rangle|$ and $-|\langle a_i, a_j\rangle|$, so $\|A_{\{i,j\}}^H A_{\{i,j\}} - I\|_{2\to 2} = |\langle a_i, a_j\rangle|$.

Taking max over $1 \leq i \neq j \leq N$, is $\boxed{\delta_2 = \mu.}$

$\delta_s \leq \mu_1(s-1) \leq (s-1)\mu$ follows from Thm. 5.3

$\left[(1-\mu_1(s-1))\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\mu_1(s-1))\|x\|_2^2 \quad \forall\ s\text{-sparse } x \in \mathbb{C}^N. \right]$ ☐

Prop. 6.3 Let $u, v \in \mathbb{C}^N$, $\|u\|_0 \leq s$, $\|v\|_0 \leq t$. If $supp(u) \cap supp(v) = \emptyset$, then

$$|\langle Au, Av\rangle| \leq \delta_{s+t} \|u\|_2 \cdot \|v\|_2$$

Proof: Let $S \triangleq supp(u) \cup supp(v)$.

$u_S, v_S \in \mathbb{C}^{|S|}$ be the restrictions of $u, v \in \mathbb{C}^N$ to S. $\langle u_S, v_S\rangle = 0$

$\because$ the supports are disjoint.

$$|\langle Au, Av\rangle| = |\langle A_S u_S, A_S v_S\rangle - \langle u_S, v_S\rangle|$$

$$= |\langle (A_S^H A_S - I) u_S, v_S \rangle|$$

$$\leq \|(A_S^H A_S - I) u_S\|_2 \cdot \|v_S\|_2 \qquad [\text{C.S.}]$$

$$\leq \underbrace{\|A_S^H A_S - I\|_{2\to 2}}_{\leq \delta_{s+t} \text{ by defn.}} \underbrace{\|u_S\|_2}_{\|u\|_2} \cdot \underbrace{\|v_S\|_2}_{\|v\|_2}$$

$$\Rightarrow |\langle Au, Av \rangle| \leq \delta_{s+t} \|u\|_2 \cdot \|v\|_2$$ as desired. ☒

Defn. 6.4 The (s,t)-restricted orthogonality const. $\theta_{s,t} = \theta_{s,t}(A)$ of $A \in \mathbb{C}^{m \times N}$ is the smallest $\theta \geq 0$ s.t.

$$|\langle Au, Av \rangle| \leq \theta \|u\|_2 \|v\|_2$$

for all disjointly supported s-sparse and t-sparse $u, v \in \mathbb{C}^N$. Equivalently,

$$\theta_{s,t} = \max\left\{ \|A_T^H A_S\|_{2\to 2},\ S \cap T = \emptyset,\ |S| \leq s,\ |T| \leq t \right\}.$$

Exercise: establish the above equivalence.

Prop. 6.5 The RIC and ROC are related by

$$\underbrace{\theta_{s,t} \leq \delta_{s+t}}_{\text{Prop. 6.3}} \leq \frac{1}{s+t}\left(s\delta_s + t\delta_t + 2\sqrt{st}\,\theta_{s,t}\right)$$

In the special case $t = s$,

$$\theta_{s,s} \leq \delta_{2s} \leq \delta_s + \theta_{s,s}$$